# 2 Sets, Functions, Sequences, and Sums

## 2.1 Sets

1. A set is a collection of objects.
   The objects of the set are called elements or members.
   Use capital letters : $A, B, C, S, X, Y$ to denote the sets.
   Use lower case letters to denote the elements: $a, b, c, x, y$.
   If $x$ is an element of the set $X$, we write $x \in X$.
   If $x$ is not an element of the set $X$, we write $x \notin X$.

2. Describing a set

   (a) list all elements if the set consists of a small number of elements:
   $X = \{a, b, c\}$
   $A = \{1, 2, \ldots, 100\}$ – need to list the first two elements to see the pattern
   $S = \{1, 3, 5, \ldots\}$ – list the first 3 elements to give away the pattern.(Not correct to list: $S = \{1, 3, 5, 7, \ldots\}$, which is redundant, nor $S = \{1, 3, \ldots\}$ because of not enough information.)
   NOTE:
   - $A = \{1, 2, 3\} = \{2, 1, 3\} = \{1, 1, 3, 2\}$
   - $\emptyset = \{\}$ is the empty set
   - $Y = \{\emptyset\} \neq \emptyset$

   (b) A set with condition(s): $S = \{x|p(x)\}$ or $\{x : p(x)\}$, that is: $S$ contains all the elements $x$ that satisfy the condition (oe have the property) $p(x)$ – a property that depends on $x$.
   Ex: $A = \{x : x \text{ is even }\} = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$.
   $S = \{x : (x - 1)(x + 2) = 0\} = \{1, -2\}$
   $T = \{x : |x| = 1\} = \{-1, 1\}$
   $X = \{x : x \text{ is a student in MA1025 }\}$.
   A more complex example: Let $A = \{1, 2, \ldots, 10\}$. Then define $B = \{x \in A : x < 7\} = \{1, 2, 3, 4, 5, 6\}$

3. Special sets
   $\mathbb{N} = \{1, 2, \ldots, \}$ is the set of all positive whole numbers
   $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the set of integers (whole numbers)
   $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$ is the set of the rational numbers

   $\mathbb{I}$ = the set of irrationals, for example: $\pi, \sqrt{2}, -\sqrt{3}$
   $\mathbb{R}$ = the real numbers
   $\mathbb{C}$ = the set of complex numbers: $a + bi$

4. We say that a set $A$ is a subset of a set $B$ if every element of $A$ is an element of $B$.
   If $A$ is a subset of $B$, we write $A \subseteq B$.

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}.$$

If a set $A$ is <u>not</u> a subset of a set $B$, we write $A \not\subseteq B$. In this case, there is an element in the set $A$ that is not in $B$.
The empty set $\emptyset$ is a subset of every set. (vacuous proof)

5. Two sets $A$ and $B$ are <u>equal</u> if $A \subseteq B$ and $B \subseteq A$. We then write $A = B$. Note that $A$ and $B$ will have the same elements, but they might be expressed differently. If they are not equal then we write $A \neq B$ (and that means that either $A$ has an element that is not in $B$, or that $B$ has an element that is not in $A$).

6. For a set $A$, we say that $S$ is a <u>proper subset</u> of a set $B$ if $A \subseteq B$ and $A \neq B$, and it is denoted by $A \subset B$.

7. For a set $S$, <u>the cardinality of $S$,$|S|$,</u> is the number of elements in the set $S$. If the cardinality is a finite number, then $S$ is said to be <u>finite</u>. Otherwise it is <u>infinite</u>. The set of natural numbers is an example of an infinite set.

8. the <u>intervals</u> are infinite sets, as described below. Let $a, b \in \mathbb{R}$
$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b.\}$
$[a, b) = \{x \in \mathbb{R} : a \leq x < b.\}$
$(a, b] = \{x \in \mathbb{R} : a < x \leq b.\}$
$(a, b) = \{x \in \mathbb{R} : a < x < b.\}$
$(a, \infty) = \{x \in \mathbb{R} : a < x.\}$
$(-\infty, b] = \{x \in \mathbb{R} : x \leq b.\}$

9. For a set $A$, the <u>power set $P(A)$ of $A$</u> is the set of all subsets of $A$.

   Ex 1: $A = \{a, b\}$. Then $\mathcal{P}(\mathcal{A}) = \{\emptyset, \{\dashv\}, \{\lfloor\}, \{\dashv, \lfloor\}\} \neq \{, \dashv, \lfloor, \{\dashv, \lfloor\}\}$ since $a, b$ are not sets without the curly braces.
   $\mathcal{P}(\mathcal{A}) = \triangle = \in^{\in} = \in^{|\mathcal{A}|}$.

   Ex 2: $C = \{\emptyset, \{\emptyset\}\}$. Then $\mathcal{P}(\mathcal{B}) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$.
   $\mathcal{P}(\mathcal{B}) = \triangle = \in^{\in} = \in^{|\mathcal{C}|}$.

10. In general: $|\mathcal{P}(\mathcal{A})| = \in^{|\mathcal{A}|}$.

11. The<u>Cartesian product</u> of two sets $A$ and $B$ is

$$A \times B = \{(a, b) : x \in A and b \in B\}.$$

   Note that $(a, b)$ is an ordered pair!! That is $(a, b) \neq (b, a)$
   Example: Let $A = \{x, y\}$ and $B = \{1, 2, 3\}$. Then

$$A \times B = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\}$$

$$B \times A = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}.$$

   Note that $A \times B \neq B \times A$.

12. What is $|A \times B|$? $|A \times B| = |A| \times |B| = 6$ in this case.

13. If $A = \emptyset$, then $A \times B = \emptyset$ and $B \times A = \emptyset$, for any set $B$.

14. the <u>truth set</u> of a predicate $P$ is the set of elements (in the given domain) that makes $P$ true.

## 2.2 Set Operations

1. When we talk about subsets, we are concerned with subsets of a larger set, usually called universal set denoted by $U$.

   We can use Venn Diagrams to represent a set:

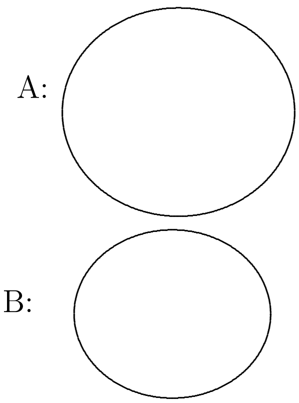


Figure 1: A Venn Diagram

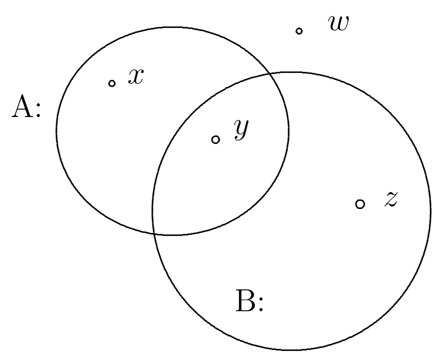


Figure 2: A Venn Diagram

   From the digram: $x \in A, y \in B, z \in A, z \in B, w \notin A, w \notin B$.

2. Let $A$ and $B$ be two sets. The following are ways of combining two or more sets:

   (a) The intersection of $A$ and $B$: $A \cap B = \{x : x \in A \text{ and } x \in B\}$. If $A \cap B = \emptyset$, then $A$ and $B$ are disjoint.

   (b) The union of $A$ and $B$: $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

   (c) The difference of $A$ and $B$: $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$.

   (d) The complement of $A$: $\bar{A} = \{x : x \notin A\} = U \setminus A$, where $U$ is the universal set.

   (e) The relative complement of $B$ in $A$: $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$.

   Example: Let $A = \{1, 3, 5, 6, 7\}$, $B = \{1, 3, 8\}$, and the universal set $U = \{1, 2, \ldots, 10\}$. What are the intersection, union, difference...?

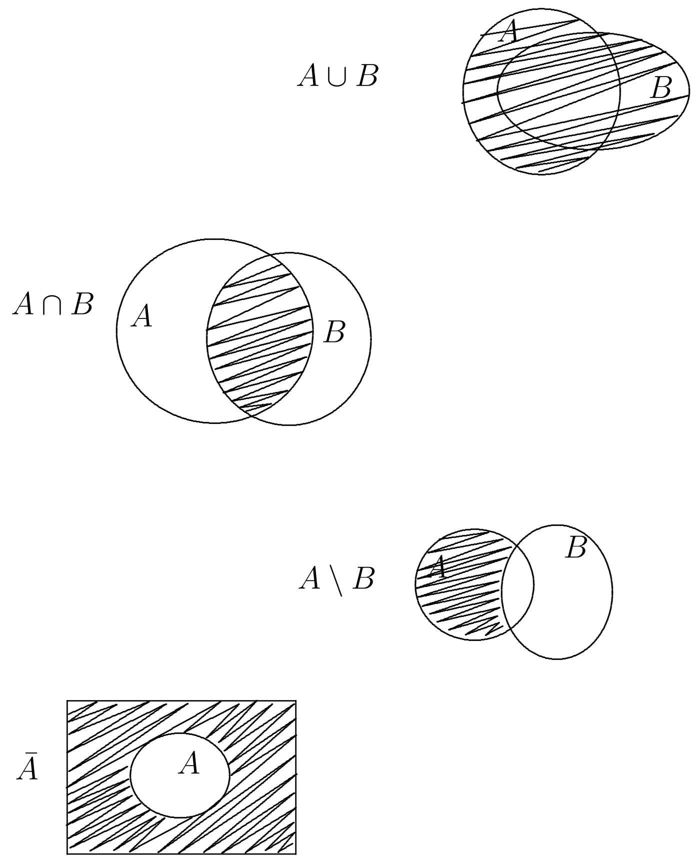


Figure 3: Set operations

(a) $A \cap B = \{1, 3\}$.
(b) $A \cup B = \{1, 3, 5, 6, 7, 8\}$.
(c) the difference $A \setminus B = \{5, 6, 7\}$, and $B \setminus A = \{8\}$
(d) $\bar{A} = \{2, 4, 8, 9, 10\}$
(e) the relative complement $A \setminus B = \{5, 6, 7\}$.

3. set identities -page 124 (note that they are similar to the "or" and "and" tables for predicates)

4. $|A \cup B| = |A| + |B| + |A \cap B|$, which is the Inclusion Exclusion principle for two sets.

5. when proving inequalities, there are three choices of techniques:
   - "chasing the element" (see Example 10 page 125): In order to show that some set $X$ is a subset of $Y$, we choose an arbitrary element $x \in X$, and we show that $x \in Y$ (where $X$ and $Y$ could be expressions involving some sets, so for Example 10, $X = \overline{A \cap B}$, and $Y = \overline{A} \cup \overline{B}$)
   - "logical equivalences" (see Example 11 page 125): Use the definition to show the inequality in question
   - "using the laws on page 124" (see Example 14 page 126): Use the laws to show the inequality in question

- "membership table" (See Example 13 page 126): This is like a truth table: you consider all the choices of $A, B$, and $C$, where $x$ could be an element of each or not.

6. Generalized Intersection and Unions: Indexed Collection of Sets
   Suppose that $A_1, A_2, \ldots, A_n$ is a collection of collection of sets, ($n \geq 3$). The following are ways of combining two or more sets:

   (a) The intersection of the $n$ sets $A_1, A_2, \ldots, A_n$ is:

   $$\bigcap_{i=1}^{n} A_i = \{x : x \in A_i, \forall i, 1 \leq i \leq n\}.$$

   (b) The union of of the $n$ sets $A_1, A_2, \ldots, A_n$ is:

   $$\bigcup_{i=1}^{n} A_i = \{x : x \in A_i, \exists i, 1 \leq i \leq n\}.$$

   Example: Let $A_i = \{i, i+1\}$, $1 \leq i \leq 10$. What are the intersection and the union of them.

   (a) $\displaystyle\bigcap_{i=1}^{10} A_i = \emptyset$.

   (b) $\displaystyle\bigcup_{i=1}^{10} A_i = \{1, 2, \ldots, 11\}$.

   Note: If we have different index sets, we have different results:
   Let $A_i = \{i, i+1\}$, and the index set $I = \{1, 5, 10\}$. Then

   (a) $\displaystyle\bigcap_{i \in I} A_i = \emptyset$.

   (b) $\displaystyle\bigcup_{i \in I} A_i = \{1, 2, 5, 6, 10, 11\}$.